COMP3411 Tutorial - Week 5
Logic

Question 1 - Propositional Logic

Decide whether each of the following sentences is valid, satisfiable, or unsatisfiable. Verify your decisions using truth tables or logical equivalence and inference rules. For those that are satisfiable, list all the models that satisfy them.

a. Smoke $\Rightarrow$ Smoke

Valid [implication, excluded middle]

b. Smoke $\Rightarrow$ Fire Satisfiable

Smoke	Fire	Smoke ⇒ Fire
T	T	T
T	F	F
F	T	T
F	F	T

Models are: {Smoke, Fire}, {Fire}, {}.

c. (Smoke $\Rightarrow$ Fire) $\Rightarrow$ (¬ Smoke $\Rightarrow$ ¬ Fire) Satisfiable

Smoke	Fire	Smoke ⇒ Fire	¬ Smoke ⇒ ¬ Fire	KB
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Models are: {Smoke, Fire}, {Smoke}, {}

d. Smoke ∨ Fire ∨ ¬ Fire

Valid

e. ((Smoke ∧ Heat) ⇒ Fire) ⇔ ((Smoke ⇒ Fire) ∨ (Heat ⇒ Fire))

Valid

((S ∧ H) ⇒ F) ⇔ (F ∨ ¬(S ∧ H)) [implication]

⇔ (F ∨ ¬ S ∨ ¬ H) [de Morgan]

⇔ (F ∨ ¬ S ∨ F ∨ ¬ H) [idempotent, commutativity]

⇔ (S ⇒ F) ∨ (H ⇒ F) [implication]

f. (Smoke ⇒ Fire) ⇒ ((Smoke ∧ Heat) ⇒ Fire)

Valid

(S ⇒ F) ⇔ (F ∨ ¬ S) [implication]

⇒ (F ∨ ¬ S ∨ ¬ H) [generalization]

⇒ (F ∨ ¬ (S ∧ H)) [de Morgan]

⇒ ((S ∧ H) ⇒ F) [conditional]

g. Big ∨ Dumb ∨ (Big ⇒ Dumb)

Valid

Big ∨ Dumb ∨ (Big ⇒ Dumb) ⇔ Big ∨ Dumb ∨ Dumb ∨ ¬ Big [implication]

⇔ Big ∨ ¬ Big ∨ Dumb [idempotent]

⇔ TRUE ∨ Dumb [excluded middle]

⇔ TRUE

h. (Big ∧ Dumb) ∨ ¬ Dumb

Satisfiable

Big	Dumb	(Big ∧ Dumb)	(Big ∧ Dumb) ∨ ¬ Dumb
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	T

Models are: {Big, Dumb}, {Big}, {}

Question 2 - Tautologies

Determine whether the following sentences are valid (i.e. tautologies) using truth tables.

(i) $((P \lor Q) \land \neg P) \to Q$
(ii) $((P \to Q) \land \neg(P \to R)) \to (P \to Q)$
(iii) $\neg(\neg P \land P) \land P$
(iv) $(P \lor Q) \to \neg(\neg P \land \neg Q)$

(i)

P	Q	$\neg P$	$P \lor Q$	$(P \lor Q) \land \neg P$	$((P \lor Q) \land \neg P) \to Q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Last column is always true no matter what truth assignment to P and Q. Therefore $((P \lor Q) \land \neg P) \to Q$ is a tautology.

(ii) $S = ((P \to Q) \land \neg(P \to R)) \to (P \to Q)$

P	Q	R	$P \to Q$	$P \to R$	$\neg(P \to R)$	$(P \to Q) \land \neg(P \to R)$	S
T	T	T	T	T	F	F	T
T	T	F	T	F	T	T	T
T	F	T	F	T	F	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	F	F	T
F	T	F	T	T	F	F	T
F	F	T	T	T	F	F	T
F	F	F	T	T	F	F	T

Last column is always true no matter what truth assignment to P, Q and R. Therefore $((P \to Q) \land \neg(P \to R)) \to (P \to Q)$ is a tautology.

(iii)

P	$\neg P$	$\neg P \land P$	$\neg(\neg P \land P)$	$\neg(\neg P \land P) \land P$
T	F	F	T	T
F	T	F	T	F

Last column is not always true. Therefore $\neg(\neg P \land P) \land P$ is not a tautology.

(iv) $(P \lor Q) \to \neg(\neg P \land \neg Q)$

P	Q	$\neg P$	$\neg Q$	$P \lor Q$	$\neg P \land \neg Q$	$\neg(\neg P \land \neg Q)$	$(P \lor Q) \to \neg(\neg P \land \neg Q)$
T	T	F	F	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	F	T	T	F	T	F	T

Last column is always true no matter what truth assignment to P and Q. Therefore $(P \lor Q) \to \neg(\neg P \land \neg Q)$ is a tautology.

Question 3 - Entailment

Show using the truth table method that the corresponding inferences are valid.

(i) $P \to Q, \neg Q \models \neg P$
(ii) $P \to Q \models \neg Q \to \neg P$
(iii) $P \to Q, Q \to R \models P \to R$

(i)

P	Q	$P \to Q$	$\neg Q$	$\neg P$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

In all rows where both $P \to Q$ and $\neg Q$ true, $\neg P$ is true. Therefore, valid inference.

(ii)

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

In all rows where both $P \to Q$ true, $\neg Q \to \neg P$ is true. Therefore, valid inference.

(iii)

P	Q	R	$P \to Q$	$Q \to R$	$P \to R$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

In all rows where both $P \to Q$ and $Q \to R$ true, $P \to R$ is true. Therefore, valid inference.

Question 4 - Inference Rules

(Exercise 7.2 from R & N)

Consider the following Knowledge Base of facts:

If the unicorn in mythical, then it is immortal, but if it is not mythical, then it is mortal and a mammal. If the unicorn in either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.

1. Translate the above statements into Propositional Logic.

   Myth ⇒ ¬ Mortal
   ¬Myth ⇒ (Mortal ∧ Mammal)
   ¬Mortal ∨ Mammal ⇒ Horned
   Horned ⇒ Magic

2. Convert this Knowledge Base into Conjunctive Normal Form.

(¬Myth ∨ ¬ Mortal) ∧ (Myth ∨ Mortal) ∧ (Myth ∨ Mammal) ∧ (Mortal ∨ Horned) ∧ (¬Mammal ∨ Horned) ∧ (¬Horned ∨ Magic)

3. Use a series of resolutions to prove that the unicorn is Horned.

Using Proof by Contradiction, we add to the database the negative of what we are trying to prov

¬Horned

We then try to derive the "empty clause" by a series of Resolutions:

<u>¬Horned ∧ (Mortal ∨ Horned)</u>
Mortal

<u>¬Horned ∧ (¬Mammal ∨ Horned)</u>
¬Mammal

<u>Mortal ∧ (¬Myth ∨ ¬Mortal)</u>
¬Myth

<u>¬Myth ∧ (Myth ∨ Mammal)</u>
Mammal

<u>Mammal ∧ ¬Mammal</u>

Having derived the empty clause, the proof (of Horned) is complete.

4. Give all models that satisfy the Knowledge Base. Can you prove that the unicorn is Mythical? How about Magical?

Because of the rule (Horned ⇒Magic), Magic must also be True. We can construct a truth table for the remaining three variables:

Myth	Mortal	Mammal	Myth ⇒ ¬ Mortal	¬ Myth ⇒ (Mortal ∧ Mammal)	KB
T	T	T	F	T	F
T	T	F	F	T	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	T	F	F

There are three models which satisfy the entire Knowledge Base:
{Horned, Magic, Myth, Mammal}, {Horned, Magic, Myth}, {Horned, Magic, Mortal, Mammal}.
We cannot prove that the unicorn is Mythical, because of the third model where Mythical is False.

Question 5 - First Order Logic

Represent the following sentences in first-order logic, using a consistent vocabulary.

a. Some students studied French in 2016.

$\exists x\ Student(x) \wedge Study(x,French,2016)$

b. Only one student studied Greek in 2015.

$\exists x\ Study(x,Greek,2015) \wedge \forall y\ (Study(y,Greek,2015) \Rightarrow y = x\ )$

sometimes written as

$\exists! x\ Study(x,Greek,2015)$

c. The highest score in Greek is always higher than the highest score in French.

$\forall t\ \exists x\ \forall y\ Score(x,Greek,t) > Score(y,French,t)$

d. Every person who buys a policy is smart.

$\forall x,p\ Person(x) \wedge Policy(p) \wedge Buy(x,p) \Rightarrow Smart(x)$

e. No person buys an expensive policy.

$\neg\exists x,p\ Person(x) \wedge Policy(p) \wedge Expensive(p) \wedge Buy(x,p)$

f. There is a barber who shaves all men in town who do not shave themselves.

$\exists b\ Barber(b) \wedge \forall m\ (Man(m) \wedge InTown(m) \wedge \neg Shave(m,m) \Rightarrow Shave(b,m))$

g. Politicians can fool some of the people all of the time, and they can fool all of the people some of the time, but they can't fool all of the people all of the time.

$\forall p\ (\ Politician(p) \Rightarrow ((\exists x \forall t\ Fool(p,x,t)) \wedge (\exists t \forall x\ Fool(p,x,t)) \wedge (\neg \forall x \forall t\ Fool(p,x,t))))$